

Kur. Myrzakul¹ and R. Myrzakulov²

Department of Mathematics, Bilkent University, 06533 Ankara, Turkey

Abstract

The connection between multidimensional soliton equations and three-dimensional Riemann space is discussed.

Contents

1	Introduction	1
2	Three-dimensional Riemann space	1
3	Integrable reductions	4
3.1	The Ishimori equation	4
3.2	The Davey-Stewartson equation	5
4	Diagonal metrics	5
4.1	$H_1 = H_2 = H, \quad H_3 = 1$	7
4.2	$H_1 = \cos \theta, \quad H_2 = \sin \theta, \quad H_3 = 1$	7
4.3	$H_1 = \cos \theta, \quad H_2 = \sin \theta, \quad H_3 = \theta_t$	7
4.4	$H_1 = e^\psi, \quad H_2 = e^\psi, \quad H_3 = \psi_t$	7
4.5	$H_1 = H_2 = H_3 = H^2$	7
5	Connections with the other equations	8
5.1	Equation (175) and the Bogomolny equation	8
5.2	Equation (175) and the Self-Dual Yang-Mills equation	8
5.3	Equation (175) and the Chern-Simons equation	9
5.4	Equation (175) as some generalization of the Lamé equation	9
6	On Lax representation of the equation (175)	10
7	Conclusion	10
8	Acknowledgments	11
9	Appendix: MISSs	11

1 Introduction

As well-known the relation between geometry and (1+1)-dimensional solitonic equations is described (the geometrical part) in terms of two-dimensional surfaces (and/or curves that is equivalent in 1+1) in a 3-dimensional M^3 (Euclidean or pseudo-Euclidean) spaces [1-24]. In 2+1 dimensions the situation is more complicated [25-30, 36-38, 40]. Usually, in this case uses the same approach, i.e., two-dimensional surfaces arbitrarily embedded in 3-dimensional space M^3 but considering also the motion (deformation) of surfaces. Of course, no doubt that it is correct but in some sense is the artificial construction.

¹Permanent address: Institute of Mathematics, Alma-Ata, Kazakhstan

²Permanent address: Institute of Physics and Technology, 480082, Alma-Ata-82, Kazakhstan. E-mail: cnlpmyra@satsun.sci.kz

Sometimes we think that in 2+1 dimensions may be the more natural geometry (at least than the above mentioned surfaces approach) is 3-dimensional Riemann spaces (see, e.g., refs.[26-29, 46-48]). This line we realized in our notes [49-51].

This note is a sequel to the preceding notes [49-51]. Our main goal in this note is to discuss the interrelation between the intrinsic geometry of three-dimensional Riemann spaces and integrable systems, in particular, spin systems in 2+1 dimensions without entering into details (for details see e.g. refs.[49-51]). We will do this without any reference to the enveloping spaces (see, also, ref. [26]). We will show that three-dimensional Riemann spaces is important not only in the general relativity but is also important in the theory of multidimensional integrable systems. As well known, the theories of multidimensional integrable systems and three-dimensional Riemann spaces are not as complete as their counterparts, respectively, integrable systems in 1+1 dimensions and two-dimensional surfaces. So, the study of the connections between integrable systems in 2+1 dimensions and three-dimensional Riemann space is one of actual problems of modern mathematical physics.

2 Three-dimensional Riemann space

Let V^3 be the space endowed with the affine connection. In this space we introduce two systems of coordinates: $(x) = (x^1, x^2, x^3)$ and $(y) = (y^1, y^2, y^3)$. It is well known from the classical differential geometry that these coordinate systems are connected by the following set of equations of second order (remark: for convenience, in [49-51] and this note we will use the unified common numerations for formulas)

$$\frac{\partial^2 y_k}{\partial x_i \partial x_j} = \Gamma_{ij}^l(x) \frac{\partial y_k}{\partial x_l} - \Gamma_{lm}^k(y) \frac{\partial y_l \partial y_m}{\partial x_i \partial x_j} \quad (155)$$

We mention that in this case the curvature tensor

$$R_{klm}^i = \left[\frac{\partial \Gamma_m}{\partial x^l} - \frac{\partial \Gamma_l}{\partial x^m} + \Gamma_l \Gamma_m - \Gamma_m \Gamma_l \right]_k^i \quad (156)$$

has only three independent components. Let the space V^3 is flat and let the metric tensor in V^3 in the coordinate system (y) is diagonal:

$$ds^2 = \sum_{i,j=1}^3 \mu_{ij} dy^i dy^j \quad (157)$$

with $\mu_{ij} = \pm 1$. Hence follows that the cooresponding connection and Riemann's curvature tensor are equal to zero:

$$\Gamma_{jk}^i(y) = 0, \quad R_{klm}^i(y) = 0 \quad (158)$$

Let for the coordinate system (x) the metric has the form

$$ds^2 = \sum_{i,j=1}^3 g_{ij} dx_i dx_j. \quad (159)$$

As the curvature tensor has the law of transformation as the four rank tensor

$$R_{klm}^i(x) = \frac{\partial x^i}{\partial y^s} \frac{\partial y^r}{\partial x^k} \frac{\partial y^q}{\partial x^l} \frac{\partial y^p}{\partial x^m} R_{klm}^i(y) \quad (160)$$

the curvature tensor for the coordinate system (x) is also equal to zero

$$R_{klm}^i(x) = 0. \quad (161)$$

Hence, for the coordinate system (x) we get the following system of three equations

$$\frac{\partial \Gamma_m}{\partial x^l} - \frac{\partial \Gamma_l}{\partial x^m} + \Gamma_l \Gamma_m - \Gamma_m \Gamma_l = 0 \quad (162)$$

where $\Gamma_m(x)$ are matrices with components

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l}). \quad (163)$$

We note that in our case the scalar curvature is equal to zero

$$R = \sum_{i,k,l,m=1}^3 g^{il} g^{km} R_{iklm} = 0. \quad (164)$$

Now the system (155) takes the form

$$\frac{\partial^2 y^k}{\partial x^i \partial x^j} = \Gamma_{ij}^l(x) \frac{\partial y^k}{\partial x^l}. \quad (165)$$

Let $\mathbf{r} = (y^1, y^2, y^3) = \mathbf{r}(x^1, x^2, x^3)$ is the position vector and put $x^1 = x, x^2 = y, x^3 = t$. Then as follows from (165) the position vector \mathbf{r} satisfies the following set of equations

$$\mathbf{r}_{xx} = \Gamma_{11}^1 \mathbf{r}_x + \Gamma_{11}^2 \mathbf{r}_y + \Gamma_{11}^3 \mathbf{r}_t \quad (166a)$$

$$\mathbf{r}_{xy} = \Gamma_{12}^1 \mathbf{r}_x + \Gamma_{12}^2 \mathbf{r}_y + \Gamma_{12}^3 \mathbf{r}_t \quad (166b)$$

$$\mathbf{r}_{xt} = \Gamma_{13}^1 \mathbf{r}_x + \Gamma_{13}^2 \mathbf{r}_y + \Gamma_{13}^3 \mathbf{r}_t \quad (166c)$$

$$\mathbf{r}_{yy} = \Gamma_{22}^1 \mathbf{r}_x + \Gamma_{22}^2 \mathbf{r}_y + \Gamma_{22}^3 \mathbf{r}_t \quad (166d)$$

$$\mathbf{r}_{yt} = \Gamma_{23}^1 \mathbf{r}_x + \Gamma_{23}^2 \mathbf{r}_y + \Gamma_{23}^3 \mathbf{r}_t \quad (166e)$$

$$\mathbf{r}_{tt} = \Gamma_{33}^1 \mathbf{r}_x + \Gamma_{33}^2 \mathbf{r}_y + \Gamma_{33}^3 \mathbf{r}_t. \quad (166f)$$

We can rewrite the equation (166) in the following form

$$Z_x = A_1 Z, \quad Z_y = A_2 Z, \quad Z_t = A_3 Z \quad (167)$$

where

$$Z = (\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_t)^T \quad (168)$$

and

$$A_1 = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & \Gamma_{11}^3 \\ \Gamma_{12}^1 & \Gamma_{12}^2 & \Gamma_{12}^3 \\ \Gamma_{13}^1 & \Gamma_{13}^2 & \Gamma_{13}^3 \end{pmatrix} \quad A_2 = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 & \Gamma_{12}^3 \\ \Gamma_{22}^1 & \Gamma_{22}^2 & \Gamma_{22}^3 \\ \Gamma_{23}^1 & \Gamma_{23}^2 & \Gamma_{23}^3 \end{pmatrix} \\ A_3 = \begin{pmatrix} \Gamma_{13}^1 & \Gamma_{13}^2 & \Gamma_{13}^3 \\ \Gamma_{23}^1 & \Gamma_{23}^2 & \Gamma_{23}^3 \\ \Gamma_{33}^1 & \Gamma_{33}^2 & \Gamma_{33}^3 \end{pmatrix}. \quad (169)$$

The compatibility condition of these equations are given by

$$\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} + [A_i, A_j] = 0. \quad (170)$$

For our further work it is convenient use the triad of unit vectors. We introduce these vectors by the way

$$\mathbf{e}_1 = \frac{\mathbf{r}_x}{H_1}, \quad \mathbf{e}_2 = \frac{\mathbf{r}_y}{H_2} + c_1 \mathbf{r}_x + c_2 \mathbf{r}_t, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2. \quad (171)$$

The explicit forms of c_1, c_2 given in [2] and

$$H_1 = | \mathbf{r}_x |, \quad H_2 = | \mathbf{r}_y |, \quad H_3 = | \mathbf{r}_t |. \quad (172)$$

The equations (156) for \mathbf{e}_k take the form

$$\begin{aligned} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x &= B_1 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B_2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \\ \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t &= B_3 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \end{aligned} \quad (173)$$

with

$$\begin{aligned} B_1 &= \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \beta \sigma & -\tau & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix} \\ B_3 &= \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix} \end{aligned} \quad (174)$$

where $k, \tau, \sigma, m_i, \omega_i$ are some real functions the explicit forms of which given in [?], $\beta = \mathbf{e}_1^2 = \pm 1, \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$. Again from the integrability condition of these equations we obtain the following set of equations

$$\frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i} + [B_i, B_j] = 0. \quad (175)$$

Many integrable systems in 2+1 dimensions are exact reductions of the equation (175) (see, e.g., [53]).

3 Integrable reductions

Now to find out particular integrable reductions of three-dimensional Riemann space, as in [49-51], we will use multidimensional integrable spin systems (MISSs). To this end, we assume that

$$\mathbf{e}_1 \equiv \mathbf{S} \quad (176)$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the spin vector, $\mathbf{S}^2 = \beta = \pm 1$. So, the vector \mathbf{e}_1 satisfies the some given MISS. Some comments on MISSs are in order. At present there exist several MISSs (see, e.g., Appendix). They play important role both in mathematics and physics. In this note, to find out a integrable case of three-dimensional Riemann space we will use the MISS - the Ishimori equation (IE).

3.1 The Ishimori equation

The IE has the form

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 S_{yy}) + u_y \mathbf{S}_x + u_x \mathbf{S}_y \quad (177a)$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y). \quad (177b)$$

The IE is integrable by Inverse Scattering Transform (IST) [42]. In this case, we get

$$m_1 = \partial_x^{-1} [\tau_y - \frac{1}{2\alpha^2} M'_2 u], \quad m_2 = -\frac{1}{2\alpha^2 k} M'_2 u$$

$$m_3 = \partial_x^{-1}[k_y + \frac{\tau}{2\alpha^2 k} M_2 u], \quad M_2' u = \alpha^2 u_{yy} - u_{xx} \quad (178)$$

and

$$\begin{aligned} \omega_2 &= -(k_x + \sigma\tau) - \alpha^2(m_{3y} + m_2 m_1) + m_2 u_x + \sigma u_y \\ \omega_3 &= (\sigma_x - k\tau) + \alpha^2(m_{2y} - m_3 m_1) + k u_y + m_3 u_x \\ \omega_1 &= \frac{1}{k}[\sigma_t - \omega_{2x} + \tau\omega_3]. \end{aligned} \quad (179)$$

Thus we expressed the functions m_k and ω_k by the three functions k, τ, σ and their derivatives. This means that we identified the equations (161) and (184) which define the geometry of Riemann space with the given MISS - the IE (177). In turn it means that we given off the integrable case of the three-dimensional Riemann space. Hence arises the natural question: how construct the integrable three-dimensional Riemann space using the MISS or that the same thing how find g_{ij} ?. The answer as follows. Let

$$\mathbf{r}_x^2 = H_1^2 = \beta = \pm 1. \quad (180)$$

Then we have that

$$\mathbf{r} = \partial_x^{-1} \mathbf{S} + \mathbf{r}_0(y, t) \quad (181)$$

where $\partial_x^{-1} = \int_{-\infty}^x dx$. For simplicity, we put $\mathbf{r}_0 = 0$. Now we can express the coefficients of the metric (143) by \mathbf{S} . As shown in [50], for the (2+1)-dimensional case, we have

$$\begin{aligned} g_{11} &= \mathbf{S}^2 = \pm 1, \quad g_{12} = \mathbf{S} \cdot \partial_x^{-1} \mathbf{S}_y, \quad g_{13} = \mathbf{S} \cdot \partial_x^{-1} \mathbf{S}_t \\ g_{22} &= (\partial_x^{-1} \mathbf{S}_y)^2, \quad g_{23} = (\partial_x^{-1} \mathbf{S}_y) \cdot (\partial_x^{-1} \mathbf{S}_t), \quad g_{33} = (\partial_x^{-1} \mathbf{S}_t)^2. \end{aligned} \quad (182)$$

3.2 The Davey-Stewartson equation

It is well known from the soliton theory that between integrable spin systems and NLS-type equations take place the so-called gauge and/or L-equivalences [40, 42]. From this fact and from the identification the three-dimensional Riemann space and the MISS (177) in the previous subsection follows that there exist some connections with NLS-type equations. We show it in this subsection. Let us we introduce two complex functions q, p as

$$q = a_1 e^{ib_1}, \quad p = a_2 e^{ib_2} \quad (183)$$

where a_j, b_j are real functions. Let a_k, b_k have the form

$$a_1^2 = \frac{|a|^2}{|b|^2} \left\{ \frac{k^2}{4} + \frac{|\alpha|^2}{4} (m_3^2 + m_2^2) - \frac{1}{2} \alpha_R k m_3 - \frac{1}{2} \alpha_I k m_2 \right\} \quad (184a)$$

$$b_1 = \partial_x^{-1} \left\{ -\frac{\gamma_1}{2ia_1'^2} - (\bar{A} - A + D - \bar{D}) \right\} \quad (184b)$$

$$a_2^2 = \frac{|b|^2}{|a|^2} \left\{ \frac{k^2}{4} + \frac{|\alpha|^2}{4} (m_3^2 + m_2^2) + \frac{1}{2} \alpha_R k m_3 - \frac{1}{2} \alpha_I k m_2 \right\} \quad (184c)$$

$$b_2 = \partial_x^{-1} \left\{ -\frac{\gamma_2}{2ia_2'^2} - (A - \bar{A} + \bar{D} - D) \right\} \quad (184d)$$

where

$$\begin{aligned} \gamma_1 &= i \left\{ \frac{1}{2} k^2 \tau + \frac{|\alpha|^2}{2} (m_3 k m_1 + m_2 k_y) - \right. \\ &\quad \left. \frac{1}{2} \alpha_R [k^2 m_1 + m_3 k \tau + m_2 k_x] + \frac{1}{2} \alpha_I [k(2k_y - m_{3x}) - k_x m_3] \right\} \end{aligned} \quad (185a)$$

$$\gamma_2 = -i \left\{ \frac{1}{2} k^2 \tau + \frac{|\alpha|^2}{2} (m_3 k m_1 + m_2 k_y) + \right.$$

$$\frac{1}{2}\alpha_R(k^2m_1 + m_3k\tau + m_2k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_xm_3]\}. \quad (185b)$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, q, p satisfy the DS equation [?]

$$iq_t + q_{xx} + \alpha^2 q_{yy} + vq = 0 \quad (186a)$$

$$ip_t - p_{xx} - \alpha^2 p_{yy} - vp = 0 \quad (186b)$$

$$\alpha^2 v_{yy} - v_{xx} = -2[\alpha^2(pq)_{yy} + (pq)_{xx}]. \quad (186c)$$

4 Diagonal metrics

Now we consider the case when the metric has the diagonal form, i.e.

$$ds^2 = \epsilon_1 H_1^2 dx^2 + \epsilon_2 H_2^2 dy^2 + \epsilon_3 H_3^2 dt^2 \quad (187)$$

where $\epsilon_i = \pm 1$. In this note we consider the case when $\epsilon_i = +1$. In this case, the Christoffel symbols take the form

$$\begin{aligned} \Gamma_{11}^1 &= \frac{H_{1x}}{H_1} = \beta_{11}, & \Gamma_{11}^2 &= -\frac{H_1 H_{1y}}{H_2^2} = -\frac{H_1}{H_2} \beta_{21}, & \Gamma_{11}^3 &= -\frac{H_1 H_{1t}}{H_3^2} = -\frac{H_1}{H_3} \beta_{31} \\ \Gamma_{12}^1 &= \frac{H_{1y}}{H_1} = \frac{H_2}{H_1} \beta_{21}, & \Gamma_{12}^2 &= \frac{H_{2x}}{H_2} = \frac{H_1}{H_2} \beta_{12}, & \Gamma_{12}^3 &= 0 \\ \Gamma_{13}^1 &= \frac{H_{1t}}{H_1} = \frac{H_3}{H_1} \beta_{31}, & \Gamma_{13}^2 &= 0, & \Gamma_{13}^3 &= \frac{H_{3x}}{H_3} = \frac{H_1}{H_3} \beta_{13} \\ \Gamma_{22}^1 &= -\frac{H_2 H_{2x}}{H_1^2} = -\frac{H_2}{H_1} \beta_{12}, & \Gamma_{22}^2 &= \frac{H_{2y}}{H_2} = \beta_{22}, & \Gamma_{22}^3 &= -\frac{H_2 H_{2t}}{H_3^2} = -\frac{H_2}{H_3} \beta_{32} \\ \Gamma_{23}^1 &= 0, & \Gamma_{23}^2 &= \frac{H_{2t}}{H_2} = \frac{H_3}{H_2} \beta_{32}, & \Gamma_{23}^3 &= \frac{H_{3y}}{H_3} = \frac{H_2}{H_3} \beta_{23} \\ \Gamma_{33}^1 &= -\frac{H_3 H_{3x}}{H_1^2} = -\frac{H_3}{H_1} \beta_{13}, & \Gamma_{33}^2 &= -\frac{H_3 H_{3y}}{H_2^2} = -\frac{H_3}{H_2} \beta_{23}, & \Gamma_{33}^3 &= \frac{H_{3t}}{H_3} = \beta_{33} \end{aligned}$$

or

$$\Gamma_{ij}^k = 0 \quad i \neq j \neq k \quad (188a)$$

$$\Gamma_{il}^i = \frac{H_{i,l}}{H_i} = \frac{H_l}{H_i} \beta_{li} \quad (188b)$$

$$\Gamma_{ll}^i = -\frac{H_l H_{li}}{H_i^2} = -\frac{H_l}{H_i} \beta_{il}, \quad i \neq l. \quad (188c)$$

Here

$$\beta_{ik} = \frac{H_{k,i}}{H_i} \quad (189)$$

are the so-called rotation coefficients. In this case we get

$$\tau = m_2 = \omega_3 = 0 \quad (190)$$

and the matrices B_i take the form

$$B_1 = \begin{pmatrix} 0 & -\beta_{21} & -\beta_{31} \\ \beta_{21} & 0 & 0 \\ \beta_{31} & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \beta_{12} & 0 \\ -\beta_{12} & 0 & -\beta_{32} \\ 0 & \beta_{32} & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & \beta_{23} \\ -\beta_{13} & -\beta_{23} & 0 \end{pmatrix}. \quad (191)$$

So we have

$$\beta_{23x} = \beta_{13}\beta_{21}, \quad \beta_{32x} = \beta_{12}\beta_{31} \quad (192a)$$

$$\beta_{13y} = \beta_{12}\beta_{23}, \quad \beta_{31y} = \beta_{32}\beta_{21} \quad (192b)$$

$$\beta_{12t} = \beta_{13}\beta_{32}, \quad \beta_{21t} = \beta_{23}\beta_{31} \quad (192c)$$

$$\beta_{12x} + \beta_{21y} + \beta_{31}\beta_{32} = 0 \quad (192d)$$

$$\beta_{13x} + \beta_{31t} + \beta_{21}\beta_{23} = 0 \quad (192e)$$

$$\beta_{23y} + \beta_{32t} + \beta_{12}\beta_{13} = 0 \quad (192f)$$

or

$$\frac{\partial \beta_{ij}}{\partial x^k} = \beta_{ik}\beta_{kj}, \quad i \neq j \neq k \quad (192g)$$

$$\frac{\partial \beta_{ij}}{\partial x^i} + \frac{\partial \beta_{ji}}{\partial x^j} + \sum_{m \neq i,j}^3 \beta_{mi}\beta_{mj} = 0, \quad i \neq j \quad (192h)$$

This nonlinear system is the famous Lamé equation and well-known in the theory of 3-orthogonal coordinates. The problem of description of curvilinear orthogonal coordinate systems in a (pseudo-) Euclidean space is a classical problem of differential geometry. It was studied in detail and mainly solved in the beginning of the 20th century. Locally, such coordinate systems are determined by $\frac{n(n-1)}{2}$ arbitrary functions of two variables. This problem in some sense is equivalent to the problem of description of diagonal flat metrics, that is, flat metrics $g_{ij}(x) = f_i(x)\delta_{ij}$. We mention that the Lamé equation describing curvilinear orthogonal coordinate systems can be integrated by the IST [33] (see also an algebraic-geometric approach in [34]). Now we would like consider some particular cases (see, also, e.g., [26]).

4.1 $H_1 = H_2 = H, \quad H_3 = 1$

Let $H = e^\psi$. We get the following set of equations

$$\psi_{xx} + \psi_{yy} + \psi_t^2 e^{2\psi} = 0 \quad (193a)$$

$$\psi_{tx} = \psi_{ty} = 0, \quad \psi_{tt} + \psi_t^2 = 0. \quad (193b)$$

From (159) we obtain $\psi_t = \frac{1}{C-t}$. So Eq. (193) reduced to the equation

$$\psi_{xx} + \psi_{yy} + \frac{1}{C-t} e^{2\psi} = 0. \quad (194)$$

4.2 $H_1 = \cos \theta, \quad H_2 = \sin \theta, \quad H_3 = 1$

In this case the corresponding equation has the form

$$(\theta_t \cos \theta)_x = -\theta_x \theta_t \sin \theta, \quad (\theta_t \sin \theta)_y = \theta_y \theta_t \cos \theta \quad (195a)$$

$$\theta_{xt} = \theta_{yt} = 0, \quad \theta_{xx} - \theta_{yy} - \theta_t^2 \sin \theta \cos \theta = 0 \quad (195b)$$

$$(\theta_t \sin \theta)_t = (\theta_t \cos \theta)_t = 0. \quad (195c)$$

$$4.3 \quad H_1 = \cos \theta, \quad H_2 = \sin \theta, \quad H_3 = \theta_t$$

In this case the corresponding equation looks like

$$\left(\frac{\theta_{ty}}{\sin \theta}\right)_x = -\frac{\theta_{xt}\theta_y}{\cos \theta} \quad (196a)$$

$$\left(\frac{\theta_{xt}}{\cos \theta}\right)_y = \frac{\theta_x\theta_{ty}}{\sin \theta} \quad (196b)$$

$$\theta_{xx} - \theta_{yy} - \sin \theta \cos \theta = 0 \quad (196c)$$

$$\left(\frac{\theta_{xt}}{\cos \theta}\right)_y - (\sin \theta)_t - \frac{\theta_y\theta_{ty}}{\sin \theta} = 0 \quad (196d)$$

$$\left(\frac{\theta_{ty}}{\sin \theta}\right)_y + (\cos \theta)_t + \frac{\theta_x\theta_{xt}}{\cos \theta} = 0. \quad (196e)$$

$$4.4 \quad H_1 = e^\psi, \quad H_2 = e^\psi, \quad H_3 = \psi_t$$

In this case we have

$$(\psi_{ty}e^{-\psi})_x = \psi_{tx}\psi_y e^{-\psi}, \quad (\psi_{tx}e^{-\psi})_y = \psi_{ty}\psi_x e^{-\psi} \quad (197a)$$

$$\psi_{xx} + \psi_{yy} + e^{2\psi} = 0 \quad (197b)$$

$$(\psi_{tx}e^{-\psi})_x + \psi_t e^\psi + \psi_y \psi_{ty} e^{-\psi} = 0, \quad (\psi_{ty}e^{-\psi})_y + \psi_t e^\psi + \psi_x \psi_{tx} e^{-\psi} = 0. \quad (197c)$$

$$4.5 \quad H_1 = H_2 = H_3 = H^2$$

In this choose we get ($H = e^\psi$)

$$\psi_{xy} = \psi_x \psi_y, \quad \psi_{xt} = \psi_x \psi_t, \quad \psi_{yt} = \psi_t \psi_y \quad (198a)$$

$$\psi_{xx} + \psi_{yy} + 4\psi_t^2 = 0, \quad \psi_{xx} + \psi_{tt} + 4\psi_y^2 = 0, \quad \psi_{tt} + \psi_{yy} + 4\psi_x^2 = 0. \quad (198b)$$

5 Connections with the other equations

It is remarkable that the equation (175) [= (162) = (170) = (213)] is related with the some well-known equations. In this section we present some of these connections.

5.1 Equation (175) and the Bogomolny equation

Consider the Bogomolny equation (BE) [31]

$$\Phi_t + [\Phi, B_3] + B_{1y} - B_{2x} + [B_1, B_2] = 0 \quad (199a)$$

$$\Phi_y + [\Phi, B_2] + B_{3x} - B_{1t} + [B_3, B_1] = 0 \quad (199b)$$

$$\Phi_x + [\Phi, B_1] + B_{2t} - B_{3y} + [B_2, B_3] = 0. \quad (199c)$$

This equation is integrable and play important role in the field theories in particular in the theory of monopols. The set of equations (175) is the particular case of the BE. In fact, as $\Phi = 0$ from (199) we obtain the system (175).

5.2 Equation (175) and the Self-Dual Yang-Mills equation

Equation (175) is exact reduction of the SO(3)-Self-Dual Yang-Mills equation (SDYME)

$$F_{\alpha\beta} = 0, \quad F_{\bar{\alpha}\bar{\beta}} = 0, \quad F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0 \quad (200)$$

Here

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} + [A_\mu, A_\nu] \quad (201)$$

and

$$\begin{aligned} \frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial z} - i \frac{\partial}{\partial t}, & \frac{\partial}{\partial x_{\bar{\alpha}}} &= \frac{\partial}{\partial z} + i \frac{\partial}{\partial t}, & \frac{\partial}{\partial x_\beta} &= \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x_{\bar{\beta}}} &= \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}. \end{aligned} \quad (202)$$

In fact, if in the SDYME (200) we take

$$A_\alpha = -iB_3, \quad A_{\bar{\alpha}} = iB_3, \quad A_\beta = B_1 - iB_2, \quad A_{\bar{\beta}} = B_1 + iB_2 \quad (203)$$

and if B_k are independent of z , then the SDYME (200) reduces to the equation (175). As known that the LR of the SDYME has the form [41, 43]

$$(\partial_\alpha + \lambda \partial_{\bar{\beta}})\Psi = (A_\alpha + \lambda A_{\bar{\beta}})\Psi, \quad (\partial_\beta - \lambda \partial_{\bar{\alpha}})\Psi = (A_\beta - \lambda A_{\bar{\alpha}})\Psi \quad (204)$$

where λ is the spectral parameter satisfying the following set of the equations

$$\lambda_\beta = \lambda \lambda_{\bar{\alpha}}, \quad \lambda_\alpha = -\lambda \lambda_{\bar{\beta}}. \quad (205)$$

Apropos, the simplest solution of this set has may be the following form

$$\lambda = \frac{a_1 x_{\bar{\alpha}} + a_2 x_{\bar{\beta}} + a_3}{a_2 x_\alpha - a_1 x_\beta + a_4}, \quad a_j = \text{consts.} \quad (206)$$

From (204) we obtain the LR of the equation (175)

$$(-i\partial_t + \lambda \partial_{\bar{\beta}})\Psi = [-iB_3 + \lambda(B_1 + iB_2)]\Psi \quad (207a)$$

$$(\partial_\beta - i\lambda \partial_t)\Psi = [(B_1 - iB_2) - i\lambda B_3]\Psi. \quad (207b)$$

5.3 Equation (175) and the Chern-Simons equation

Consider the action of the Chern-Simons (CS) theory [44]

$$S[J] = \frac{k}{4\pi} \int_M \text{tr}(J \wedge dJ + \frac{2}{3} J \wedge J \wedge J) \quad (208)$$

where J is a 1-form gauge connection with values in the Lie algebra \hat{g} of a (compact or noncompact) non-Abelian simple Lie group \hat{G} on an oriented closed 3-dimensional manifold M , k is the coupling constant. The classical equation of motion is the zero-curvature condition

$$dJ + J \wedge J = 0. \quad (209)$$

Let the 1-form J has the form

$$J = B_1 dx + B_2 dy + B_3 dt. \quad (210)$$

As shown in [44], substituting the (210) into (209) we obtain the equation (175). Note that from this fact and from the results of the subsection 5.2 follows that the CS - equation of motion (209) is exact reduction of the SDYM equation (200).

5.4 Equation (175) as some generalization of the Lamé equation

Let us the matrices B_i (174) we rewrite in the form

$$B_1 = \begin{pmatrix} 0 & -\beta_{21} & -\beta_{31} \\ \beta_{21} & 0 & \tau \\ \beta_{31} & -\tau & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \beta_{12} & -m_2 \\ -\beta_{12} & 0 & -\beta_{32} \\ m_2 & \beta_{32} & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & \omega_3 & \beta_{13} \\ -\omega_3 & 0 & \beta_{23} \\ -\beta_{13} & -\beta_{23} & 0 \end{pmatrix}. \quad (211)$$

Then the equation (175) in elements takes the form

$$\beta_{23x} - \tau_t = \beta_{13}\beta_{21} - \omega_3\beta_{31} \quad (212a)$$

$$\beta_{32x} + \tau_y = \beta_{12}\beta_{31} + m_2\beta_{21} \quad (212b)$$

$$\beta_{13y} + m_{2t} = \beta_{12}\beta_{23} + \omega_3\beta_{32} \quad (212c)$$

$$\beta_{31y} - m_{2x} = \beta_{32}\beta_{21} - \tau\beta_{12} \quad (212d)$$

$$\beta_{12t} - \omega_{3y} = \beta_{13}\beta_{32} - m_3\beta_{23} \quad (212e)$$

$$\beta_{21t} + \omega_{3x} = \beta_{23}\beta_{31} + \tau\beta_{13} \quad (212f)$$

$$\beta_{12x} + \beta_{21y} + \beta_{31}\beta_{32} + \tau m_2 = 0 \quad (212g)$$

$$\beta_{13x} + \beta_{31t} + \beta_{21}\beta_{23} + \tau\omega_3 = 0 \quad (212h)$$

$$\beta_{23y} + \beta_{32t} + \beta_{12}\beta_{13} + m_2\omega_3 = 0 \quad (212i)$$

Hence as $\tau = m_2 = \omega_3 = 0$ we obtain the Lamé equation (192). So, the equation (175) is one of the generalizations of the Lamé equation.

6 On Lax representation of the equation (175)

As follows from the results of the section 3, Equation (175) can admits several integrable reductions. At the same time, the results of the subsections 5.1-5.2 show that the equation (175) is integrable may be and in general case. At least, it admits the LR of the form (207) and/or of the following form (see, e.g., [38])

$$\Phi_x = U_1\Phi, \quad \Phi_y = U_2\Phi, \quad \Phi_t = U_3\Phi \quad (213)$$

with

$$U_1 = \frac{1}{2} \begin{pmatrix} i\tau & -(k+i\sigma) \\ k-i\sigma & -i\tau \end{pmatrix}, \quad U_2 = \frac{1}{2} \begin{pmatrix} im_1 & -(m_3+im_2) \\ m_3-im_2 & -im_1 \end{pmatrix}$$

$$U_3 = \frac{1}{2} \begin{pmatrix} i\omega_1 & -(\omega_3+i\omega_2) \\ \omega_3-i\omega_2 & -i\omega_1 \end{pmatrix}. \quad (214)$$

Systems of this type were first studied by Zakharov and Shabat [35]. The integrability conditions on this system of overdetermined equations (211), require that

$$U_{i,j} - U_{j,i} + [U_i, U_j] = 0. \quad (215)$$

Many (and perhaps all) integrable systems in 2+1 dimensions have the LR of the form (211). In our case, the IE (177) and the DS equation (186) have also the LR of the form (211) with the functions

m_i, ω_i given by (123) and (124). On the other hand, it is well-known that for example the DS equation (186) has the following LR of the standard form

$$\alpha\Psi_y = \sigma_3\Psi_x + Q\Psi, \quad Q = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \quad (216a)$$

$$\Psi_t = 2i\sigma_3\Psi_{xx} + 2iQ\Psi_x + \begin{pmatrix} c_{11} & iq_x + i\alpha q_y \\ ip_x - i\alpha p_y & c_{22} \end{pmatrix} \Psi \quad (216b)$$

with

$$c_{11x} - \alpha c_{11y} = i[(pq)_x + \alpha(pq)_y], \quad c_{22x} + \alpha c_{22y} = -i[(pq)_x - \alpha(pq)_y]. \quad (217)$$

Hence arises the natural question: how connected the both LR for one and the same integrable systems (in our case for the DS equation)?. In fact, these two LR are related by the gauge transformation [50-51]

$$\Phi = g\Psi \quad (218)$$

where Φ and Ψ are some solutions of the equations (222) and (111), respectively, while g is the some matrix.

7 Conclusion

In this note we have studied the relation between integrable systems in 2+1 dimensions and 3-dimensional Riemann spaces. We have shown that in this geometrical setting certain typical structures of the completely integrable (2+1)-dimensional systems arise. To find out the integrable cases of the 3-dimensional Riemann space as an examples we used the Ishimori and Davey-Stewartson equations. The connections of the equations characterizing of the 3-dimensional Riemann space with the other known equations such as the Bogomolny and Self-Dual Yang-Mills equations are considered. Such connection with the Chern-Simons equation of motion was established in [?]. Of course our approach needs further developments. Finally we note that the details of some calculations were given in [50-51].

8 Acknowledgments

This work was partially supported by INTAS (grant 99-1782). RM would like to thanks to V.S.Dryuma, M.Gurses, B.G. Konopelchenko, D.Levi, L.Martina and G.Soliani for very helpful discussions and especially D.Levi for the financial support and kind hospitality. He is grateful to the EINSTEIN Consortium of Lecce University and the Department of Mathematics of Bilkent University for their financial supports and warm hospitality.

9 Appendix: MISSs

At present there exist several MISSs. Here we will give some of them.

i) *The Myrzakulov I (M-I) equation* The simplest example of MISS is the M-I (Remark: about our notations please see e.g., ref. [38]) equation looks like

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x \quad (219a)$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) \quad (219b)$$

ii) *The Myrzakulov VIII (M-VIII) equation.* The M-VIII equation is one of simplest MISSs in 2+1 dimensions and reads as [52]

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} + u\mathbf{S}_x \quad (220a)$$

$$u_x + u_y + \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) = 0 \quad (220b)$$

iii) *The Ishimori equation.* The famous Ishimori equation has the form

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 \mathbf{S}_{yy}) + u_x \mathbf{S}_y + u_y \mathbf{S}_x \quad (221a)$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) \quad (221b)$$

iv) *The Myrzakulov IX (M-IX) equation.* This equation reads as [52]

$$\mathbf{S}_t = \mathbf{S} \wedge M_1 \mathbf{S} - iA_1 \mathbf{S}_y - iA_2 \mathbf{S}_x \quad (222a)$$

$$M_2 u = 2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) \quad (222b)$$

Here M_i, A_i have the forms

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b-a) \frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b) \frac{\partial^2}{\partial x^2}$$

$$M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(2a+1) \frac{\partial^2}{\partial x \partial y} + 4a(a+1) \frac{\partial^2}{\partial x^2}$$

$$A_1 = i\{\alpha(2b+1)u_y - 2(2ab+a+b)u_x\}$$

$$A_2 = i\{4\alpha^{-1}(2a^2b+a^2+2ab+b)u_x - 2(2ab+a+b)u_y\}$$

The M-IX equation contains several particular integrable cases: a) the M-VIII equation (218) as $a = b = -1$; b) the Ishimori equation (177) as $a = b = -\frac{1}{2}$; c) the M-XXXIV equation as $a = b = -1, y = t$ [52]

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} + u \mathbf{S}_x \quad (223a)$$

$$u_t + u_x + \frac{1}{2}(\mathbf{S}_x^2)^2 = 0 \quad (223b)$$

and so on. The M-XXXIV equation (221) describe the nonlinear dynamics of compressible magnets [45]. It is the first (and, to the best of our knowledge, at present the unique) example of integrable spin system governing the nonlinear interactions of spin (\mathbf{S}) and lattice (u) subsystems in 1+1 dimensions.

v) *The Myrzakulov XX (M-XX) equation.* This equation reads as [52]

$$\mathbf{S}_t = \mathbf{S} \wedge \{(b+1)\mathbf{S}_{xx} - b\mathbf{S}_{yy}\} + bu_y \mathbf{S}_y + (b+1)u_x \mathbf{S}_x \quad (224a)$$

$$u_{xy} = \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) \quad (224b)$$

vi) *The (2+1)-dimensional Myrzakulov 0 (M-0) equation.* The (2+1)-dimensional M-0 equation [52]

$$\mathbf{S}_t = c_1 \mathbf{S}_x + c_2 \mathbf{S}_y \quad (225)$$

is in general not integrable but admits integrable reductions. For example, the following case of the (2+1)-dimensional M-0 equation is integrable

$$\mathbf{S}_t = -\frac{\beta_{13}}{\beta_{31}} \mathbf{S}_x - \frac{\beta_{13}}{\beta_{12}\beta_{31}} \mathbf{S}_y \quad (226a)$$

$$\frac{\partial \beta_{ij}}{\partial x^k} = \beta_{ik} \beta_{kj} \quad (226b)$$

$$\frac{\partial \beta_{ij}}{\partial x^i} + \frac{\partial \beta_{ji}}{\partial x^j} + \sum_{m \neq i,j}^3 \beta_{mi} \beta_{mj} = 0 \quad (226c)$$

and so on. All of these MISSs are some (2+1)-dimensional integrable extensions of the isotropic Landau-Lifshitz (LL) equation

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} \quad (227)$$

and in 1+1 dimensions reduced to it. Here we would like mention that there exist the other classes MISSs which are not multidimensional generalizations of the LL equation (227), for example, the M-II, M-III and M-XXII equations and so on. Finally we note that all MISSs in 2+1 dimensions are the integrable particular cases of the M-0 equation (225).

References

- [1] Fokas A S and Gelfand I M, Commun. Math. Phys, **177**, 203 (1996); **205**, 37 (1995).
- [2] Gurses M, Phys.Lett. A241, 329 (1998)
- [3] Ceyhan O, Fokas A S and Gurses M, J.Math. Phys., 41, 2251 (2000)
- [4] Lamb Jr G L, Phys.Rev.Lett. **37**, 235 (1976)
- [5] Lund F and Regge T, Phys.Rev.D. **14**, 1524 (1976)
- [6] Lakshmanan M, Phys.Lett.A. **61**, 53 (1977)
- [7] Sasaki R, Nucl.Phys.B. **154**, 343 (1979)
- [8] Sym A, *Springer Lecture Notes in Physics*, ed. R.Martini (Springer, Berlin, 1985), **239**, 154 (1985)
- [9] Doliwa A and Santini R M, Phys.Lett.A. **185**, 373 (1994)
- [10] Pelce P, in *Dynamics of Curves Fronts* (Academic, New York, 1986)
- [11] Nakayama K, Segur H and Wadati M, Phys.Rev.Lett. **69**, 2603 (1992)
- [12] Balakrishnan R and Dandoloff R, Phys.Lett. A **260**, 62 (1999)
- [13] Bobenko A I, in *Harmonic maps and integrable systems* eds. A.Fordy and J.Wood. (Vieweg, 1994) 83-127 (1994)
- [14] Goldstein R E and Petrich D M, Phys.Rev.Lett. **67**, 3203 (1991)
- [15] Gurses M and Nutku Y, J.Math.Phys. **22**, 1393 (1981)
- [16] Sym A, Ragnisco O, Levi D and Brushi M, Lett. Nuovo Cimento Soc. Ital. Fis. **44**, 529 (1985)
- [17] Pinkall U and Sterling I, Ann. Math. **130**, 407 (1989)
- [18] Korotkin D A, J.Math. Sci. **94**, 65 (1999)
- [19] Tafel J, J.Geo.Phys. **17**, 381 (1995)
- [20] Levi D and Sym A, Phys.Lett. A **149**, 381 (1995)
- [21] Schief W K, Proc. R. Soc. London, Ser. A. **453**, 1671 (1997)
- [22] Cieslinski J, Goldstein P and Sym A, Phys.Lett. A **205**, 37 (1995)
- [23] Wu H, Ann.Global Anal.Geo. **11**, 49 (1993)
- [24] Cieslinski J, J.Math.Phys. **38**, 4255 (1997)
- [25] Li Y and Chen C, J.Math.Phys. **41**, 2066 (2000)
- [26] Konopelchenko B G, J.Math.Phys. **38**, 434 (1997).
- [27] Konopelchenko B G, Phys.Lett. A **183**, 153 (1993).
- [28] Dryuma V S and Konopelchenko B G, Izv. AN RM. **3**, 61 (1996)
- [29] Dryuma V S, *On the Law of Transformation of Affine Connection and its Integration. Part 1. Generalization of the Lamé equations.* (e-preprint: solv-int/9803004)

- [30] Taimanov I A, Trans. Amer.Math.Soc. Ser. 2, **179**, 133 (1997)
- [31] Ablowitz M J and Clarkson P A, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge, 1992)
- [32] Martina L, Pashaev O K and Soliani G, J.Phys. A **27**, 943 (1994)
- [33] Zakharov V E, Duke Math. J. **94**, 103 (1998)
- [34] Krichever I M, Func. Anal.Appl. **31**, 25 (1997)
- [35] Zakharov V E and Shabat A B, Func.Anal.Appl. **13**, 166 (1979)
- [36] Myrzakulov R, Vijayalakshmi S, Nugmanova G N and Lakshmanan M, Phys. Lett.A. **233**, 391 (1997)
- [37] Myrzakulov R, Vijayalakshmi S, Syzdykova R N and Lakshmanan M, J.Math.Phys. **39**, 2122 (1998)
- [38] Lakshmanan M, Myrzakulov R, Vijayalakshmi S and Danlybaeva A K, J.Math.Phys. **39**, 3765 (1998)
- [39] Myrzakulov R, Nugmanova G N and Syzdykova R N, J.Phys. A: Math.Gen. **31**, 9535 (1998)
- [40] Myrzakulov R, Nugmanova G N and Danlybaeva A K, Theor. Math. Phys. **118**, 347 (1999)
- [41] Ward R S, Phys. Lett.A. **61**, 81 (1977)
- [42] Konopelchenko B G, *Solitons in Multidimensions* (Word Scientific, Singapore,1993).
- [43] Belavin A A and Zakharov V E, Phys. Lett. B **73**, 53 (1978)
- [44] Martina L, Myrzakul Kur, Myrzakulov R and Soliani G, *Deformation of surfaces, integrable systems and Chern-Simon theory*, (e-preprint: nlin.SI/0006039)
- [45] Myrzakulov R, Daniel M and Amuda R, Physica A. **234**, 715 (1997)
- [46] Chern S S and Tenenblat K, Stud.Appl.Math. **74**, 55 (1986).
- [47] Ablowitz M, Beals R and Tenenblat K, Stud.Appl.Math. **74**, 177 (1986)
- [48] Dodd R K, Commun.Math.Phys. **197**, 641 (1998)
- [49] Myrzakul Kur, *Notes on Riemann geometry and integrable systems. Part I: The (1+1)-dimensional case.* (Preprint CNLP-1997, Almaty, 1997)
- [50] Myrzakul Kur and Myrzakulov R, *Notes on Riemann geometry and integrable systems. Part II: The (2+1)-dimensional case.* (Preprint CNLP-1998, Almaty, 1998)
- [51] Myrzakul Kur and Myrzakulov R, *Notes on Riemann geometry and integrable systems. Part III: The general formulation.* (Preprint CNLP-1999, Almaty, 1999)
- [52] Myrzakulov R, *On some integrable and nonintegrable soliton equations of magnets* (Preprints HEPI, Alma-Ata, 1987).
- [53] Myrzakulov R, *Soliton equations in 2+1 dimensions and Differential geometry of curves/surfaces* (Preprint CNLP-1994-02, Alma-Ata, 1994).